# Chapter 6 Techniques of Integration

Recall: Some important integrals that we have learnt so far.

Table of Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \tan x dx = \ln\left|\sec x\right| + C$$

$$\int \cot x dx = \ln\left|\sin x\right| + C$$

$$\int \sec x dx = \ln\left|\sec x + \tan x\right| + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$$

# Section 6.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts.*

The Product Rule states that if $f$ and $g$ are differentiable functions, then

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int\left[f(x)g'(x)+g(x)f'(x)\right]dx=f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx+\int g(x)f'(x)\,dx=f(x)g(x)$$

We can rearrange this equation as

$$\int f(x)g'(x)\,dx=f(x)g(x)-\int g(x)f'(x)\,dx$$

This formula is called the **formula for integration by parts**. It is perhaps easier to remember in the following notation. Let $u=f(x)$ and $v=g(x)$. Then the differentials are $du=f'(x)dx$ and $dv=g'(x)dx$, so, by the Substitution Rule, the formula for integration by parts becomes

$$\int u\,dv=uv-\int v\,du$$

<u>Example 1</u>: Find $\int x\cos x\,dx$.

Our aim in using integration by parts is to obtain a simpler integral than the one we started with. Thus in Example 1 we started with $\int x\cos x dx$ and expressed it in term of the simpler integral $\int \sin x dx$. If we had instead chosen $u = \cos x$ and $dv = x dx$, then $du = -\sin x\, dx$ and $v = x^2/2$, so integration by parts gives

$$\int x\cos x dx = \cos x\frac{x^2}{2} + \frac{1}{2}\int x^2\sin x dx$$

Although this is true, $\int x^2 \sin x dx$ is more difficult integral than the one we started with. In general, when deciding on a choice for $u$ and $dv$, we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated as long as $dv = g'(x)dx$ can readily integrated to give $v$.

<u>Example 2</u>: Evaluate $\int x\ln x\, dx$.

<u>Example 3</u>: Find $\int x^2 \cos \pi x\, dx$.

Example 4: Evaluate $\int e^{-x}\cos x\,dx$ .

If we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts. Assuming $f'$ and $g'$ are continuous, and using the Fundamental Theorem, we obtain

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

If we let $u = f(x)$ and $v = g(x)$, then we have

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

<u>Example 5</u>: Evaluate $\int_0^1 \cos^{-1} x\, dx$

# Section 6.2 Integrating Rational Functions by Partial Fractions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate. To illustrate the method, observe that by taking the fractions $2/(x+3)$ and $1/(x-2)$ to a common denominator we obtain

$$\frac{2}{x+3}-\frac{1}{x-2}=\frac{2(x-2)-1(x+3)}{(x+3)(x-2)}=\frac{x-7}{x^2+x-6}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\int\frac{x-7}{x^2+x-6}\,dx=\int\left(\frac{2}{x+3}-\frac{1}{x-2}\right)dx=2\ln|x+3|-\ln|x-2|+C$$

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x)=\frac{P(x)}{Q(x)}$$

where $P$ and $Q$ are polynomials. It's possible to express $f$ as a sum of simpler fractions provided that the degree of $P$ , denoted as deg($P$), is less than the degree of $Q$, denoted as deg($Q$).

If $f$ is improper, that is, $\deg(P)\ge\deg(Q)$, then we must take the preliminary step of dividing $Q$ into $P$ (by long division) until a remainder $R(x)$ is obtained such that $\deg(R)<\deg(Q)$. The division statement is

$$f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$$

where $S$ and $R$ are also polynomials.

<u>Example 1</u>: $\int\frac{x^2+6}{x+4}dx$

The next step is to factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial $Q$ can be factored as a product of linear factors (of the form $ax+b$) and irreducible quadratic factors (of the form $ax^2+bx+c$, where $b^2-4ac<0$). For instance, if $Q(x)=x^4-16$, we could factor it as

$$Q(x)=x^4-16=\left(x^2+4\right)\left(x^2-4\right)=\left(x^2+4\right)(x+2)(x-2)$$

The third step is to express the proper rational function $R(x)/Q(x)$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{\left(ax^2+bx+c\right)^j}$$

**<u>CASE I</u>: The denominator $Q(x)$ is a product of distinct linear factors.**

This means that we can write

$$Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k} \qquad (1)$$

<u>Example 2</u>: Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

**<u>CASE II</u>: $Q(x)$ is a product of linear factors, some of which are repeated.**

Suppose the first linear factor $(a_1x+b_1)$ is repeated *r* times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x+b_1)$ in Equation (1), we would use

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r} \tag{2}$$

By way of illustration, we could write

$$\frac{x^3+4}{x^2(x+1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}$$

<u>Example 3</u>: Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

**<u>CASE III</u>: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.**

If $Q(x)$ has the factor $ax^2+bx+c$, where $b^2-4ac<0$, then, in addition to the partial fractions in Equation (1) and (2), the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c} \qquad (3)$$

where $A$ and $B$ are constants to be determined. For instance, the function given by $x/\left[(x-3)\left(x^2+1\right)\left(x^2+2\right)\right]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-3)\left(x^2+1\right)\left(x^2+2\right)}=\frac{A}{x-3}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2+2}$$

The term given in (3) can be integrated by completing the square and using the formula

$$\boxed{\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$$

<u>Example 4</u>: Evaluate $\int\frac{2x^2-x+4}{x^3+4x}dx$

**<u>CASE IV</u>: $Q(x)$ contains a repeated irreducible quadratic factor.**

If $Q(x)$ has the factor $\left(ax^2+bx+c\right)^r$, where $b^2-4ac<0$, then instead of the single partial fraction (3), the sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{A_rx+B_r}{\left(ax^2+bx+c\right)^r} \qquad (4)$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the term in (4) can be integrated by first completing the square.

<u>Example 5</u>: Evaluate $\int \frac{1-x+2x^2-x^3}{x\left(x^2+1\right)^2}\,dx$

# Section 6.3 Improper Integrals

In defining a definite integral $\int_a^b f(x)\,dx$ we dealt with a function *f* defined on a finite interval [*a*, b] and we assumed that *f* does not have an infinite discontinuity. In this section we extend the concept of a definite integral to case where the interval is infinite and also to the case where *f* has an infinite discontinuity in [*a*, b]. In either case the integral is called an *improper integral*.

TYPE I: Infinite Intervals

Definition of an improper integral of type I

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \ge a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \le b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integral $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called **convergen**t if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

In part (c) any real number *a* can be used.

Example 1: Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

Example 2: Evaluate $\int_{-\infty}^{0} xe^{x} dx$ .

Example 3: Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

<u>TYPE II</u>: Discontinuous Integrands

Definition of an improper integral of type II

(a) If $f$ is continuous on $[a, b)$ and is discontinuous at $b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If $f$ is continuous on $(a, b]$ and is discontinuous at $a$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergen**t if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If $f$ has a discontinuity at $c$, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

<u>Example 4</u>: Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

Example 5: Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible

Example 6: Evaluate $\int_0^1 \ln x \, dx$